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# Asymptotics and oscillation of $n$ th-order nonlinear differential equations with $p$ -Laplacian like operators

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## Abstract

This paper is concerned with  $n$ th-order nonlinear differential equations of the form  $(a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t))' + r(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t) + q(t)|x(g(t))|^{p-2}x(g(t)) = 0$  with  $n \geq 2$ . By discussing the signs of  $i$ th-order derivatives of eventually positive solutions, for  $i = 1, \dots, n-1$ , and using the generalized Riccati technique and integral averaging technique, we derive new criteria for oscillation and asymptotic behavior of the equation. Our results generalize and improve many existing results in the literature.

**Keywords:**  $n$ th-order nonlinear differential equations; asymptotic behavior; oscillation;  $p$ -Laplacian

## 1 Introduction

In this paper, we study the  $n$ th-order nonlinear differential equation with Laplacian and deviating argument

$$(a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t))' + r(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t) + q(t)|x(g(t))|^{p-2}x(g(t)) = 0, \quad (1.1)$$

where  $t \in [t_0, \infty)$ . Throughout this paper, we assume the following:

(H1)  $a(t) \in C^1([t_0, \infty), (0, \infty))$ ,  $r(t), q(t) \in C([t_0, \infty), \mathbb{R})$ ,  $q(t) \geq 0$ , and  $a'(t) + r(t) \geq 0$ ;

(H2)  $p > 1$  is a real number,  $g(t) \in C([t_0, \infty), \mathbb{R})$  such that  $\lim_{t \rightarrow \infty} g(t) = \infty$ .

Asymptotics and oscillation of (1.1) and related equations have been discussed by many authors; see [1–16] and the references therein. In particular in 2014, Zhang *et al.* [14] established oscillation criteria for (1.1) when  $n \geq 4$  is even via the integral averaging technique and two kinds of functions  $H(t, s)$  and  $H_*(t, s)$ , employed the comparison technique to discuss the oscillation of (1.1) when  $n \geq 2$  is even and the oscillation and asymptotic behavior of (1.1) when  $n \geq 3$  is odd.

By imposing some additional assumptions, in the present paper we shall discuss the signs of  $i$ th-order derivatives of eventually positive solutions for  $i = 1, \dots, n-1$ , and we establish concrete criteria for the asymptotics and oscillation of (1.1) for both the even-order and the odd-order cases, where the deviating arguments may be retarded, advanced, or mixed. Our results will generalize and improve those in [14] and many other papers for the even-order case and develop new results for the odd-order case.

By a solution of (1.1) we mean a nontrivial real-valued function  $x(t) \in C^{n-1}([t_0, \infty), \mathbb{R})$  such that  $a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t) \in C^1([t_0, \infty), \mathbb{R})$ , which satisfies (1.1). Our attention is restricted to those solutions of (1.1) that satisfy  $\sup\{|x(t)| : t \geq t_x\} > 0$  for any  $t_x \geq t_0$ . A solution  $x(t)$  of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

This paper is organized as follows: After this introduction, we present the main results in Section 2, followed by illustrative examples in Section 3. We then introduce some preliminary lemmas in Section 4, which are used to prove the main results in Section 5. The conclusion is drawn in Section 6.

## 2 Main results

In this section, we present our main results which provide conditions for every solution of (1.1) to be oscillatory on  $[t_0, \infty)$  or convergent to 0 as  $t \rightarrow \infty$ . In order to state the main theorems, we need the following notation.

For  $t, T \in \mathbb{R}$  such that  $t \geq T$ , we define

$$\begin{aligned}\beta_0(t, T) &= \left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) \right)^{-\frac{1}{p-1}}, \\ \beta_i(t, T) &= \int_T^t \beta_{i-1}(s, T) ds, \\ \beta^*(t, T) &= \beta_{n-1}^{p-2}(t, T) \beta_{n-2}(t, T), \\ \beta_*(t, T) &= \min \{ \beta_{n-1}(t, T), \beta_{n-1}(g(t), T) \}, \\ Q(t) &= q(t)G(t), \quad G(t) = \begin{cases} \left( \frac{g^{n-1}(t)}{t^{n-1}} \right)^{p-1}, & g(t) \leq t, \\ 1, & g(t) > t. \end{cases}\end{aligned}$$

For  $D = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$ , we define

$$\begin{aligned}\mathcal{H} &= \{H(t, s) \in C^1(D, [0, \infty)) : H(t, t) = 0, H(t, s) > 0 \text{ and } H'_s(t, s) \leq 0 \text{ for } t > s \geq 0\}, \\ \left[ H'_s(t, s) + H(t, s) \left( \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right) \right]_+ &= \max \left\{ H'_s(t, s) + H(t, s) \left( \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right), 0 \right\},\end{aligned}$$

where  $z \in C^1(\mathbb{R}, (0, \infty))$  is to be given in Theorem 2.1,  $z'_+(t) = \max\{z'(t), 0\}$ , and  $\left[ \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right]_+ = \max\left\{ \left[ \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right], 0 \right\}$ .

The results in the first theorem are valid for all  $p > 1$ .

**Theorem 2.1** *Let  $p > 1$ . Assume that (H1)-(H2) and*

$$\int_{t_0}^{\infty} \left[ \frac{\exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right)}{a(s)} \right]^{\frac{1}{p-1}} ds = \infty \quad (2.1)$$

*hold, and either*

$$\int_{t_0}^{\infty} \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) q(s) ds = \infty, \quad (2.2)$$

or

$$\int_{t_0}^{\infty} \left[ \int_v^{\infty} \left( a^{-1}(s) \exp \left( - \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) \int_s^{\infty} \exp \left( \int_{t_0}^u \frac{r(\tau)}{a(\tau)} d\tau \right) q(u) du \right)^{\frac{1}{p-1}} ds \right] dv = \infty. \quad (2.3)$$

Furthermore, for sufficiently large  $T \in \mathbb{R}$ , one of the following conditions is satisfied:

(a) there exists a  $z \in C^1(\mathbb{R}, (0, \infty))$  such that

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ z(s) \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) Q(s) - \frac{z'_+(s)}{\beta_{n-1}^{p-1}(s, T)} \right] ds = \infty,$$

(b) there exists a  $z \in C^1(\mathbb{R}, (0, \infty))$  such that

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ z(s) Q(s) - \frac{1}{p^p} \left[ \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right]^p \frac{z(s)}{\beta_{n-2}^{p-1}(s, T) \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right)} \right] ds = \infty,$$

(c) there exist a  $z \in C^1(\mathbb{R}, (0, \infty))$  and an  $H \in \mathcal{H}$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) z(s) Q(s) - \frac{[H'_s(t, s) + H(t, s) \left( \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right)]^p z(s)}{p^p \beta_{n-2}^{p-1}(s, T) H^{p-1}(t, s) \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right)} \right] ds = \infty.$$

Then:

- (i) every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$  when  $n$  is odd, and
- (ii) every solution  $x(t)$  of (1.1) is oscillatory when  $n$  is even.

The results in the second theorem hold only for  $g(t) \geq t$  or  $g(t) \leq t$  with  $g'(t) > 0$ .

**Theorem 2.2** Let  $p > 1$ . Assume that (H1)-(H2) and (2.1) hold. If either (2.2) is satisfied, or (2.3) is satisfied and for sufficiently large  $T \in \mathbb{R}$ ,

$$\beta_*^{p-1}(t, T) \int_t^{\infty} \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) q(s) ds > 1, \quad (2.4)$$

then:

- (i) every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$  when  $n$  is odd, and
- (ii) every solution  $x(t)$  of (1.1) is oscillatory when  $n$  is even.

The results in the third theorem hold only for  $p \geq 2$ .

**Theorem 2.3** Let  $p \geq 2$ . Assume that (H1)-(H2) and (2.1) hold and either (2.2) or (2.3) is satisfied. Furthermore, for sufficiently large  $T \in \mathbb{R}$ , there exists a  $z \in C^1(\mathbb{R}, (0, \infty))$  such that one of the following conditions is satisfied:

(a)

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ z(s)Q(s) - \frac{\left[ \frac{z'(s)}{z(s)} - \frac{r(s)}{a(s)} \right]^2 z(s)}{4(p-1) \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) \beta^*(s, T)} \right] ds = \infty,$$

(b) *there exists an  $H \in \mathcal{H}$  such that*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s)z(s)Q(s) - \frac{[H'_s(t, s) + H(t, s)\left(\frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)}\right)]^2 z(s)}{4(p-1)\beta^*(s, T) \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) H(t, s)} \right] ds \\ = \infty. \end{aligned}$$

*Then:*

- (i) *every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$  when  $n$  is odd, and*
- (ii) *every solution  $x(t)$  of (1.1) is oscillatory when  $n$  is even.*

### 3 Examples

In this section, we give two examples to illustrate our main results.

**Example 1** Consider the equation

$$\left( \frac{1}{t} |x^{(n-1)}(t)|^{-\frac{2}{3}} x^{(n-1)}(t) \right)' + t^3 |x^{(n-1)}(t)|^{-\frac{2}{3}} x^{(n-1)}(t) + t^{-\frac{1}{3}} |x(t-1)|^{-\frac{2}{3}} x(t-1) = 0, \quad (3.1)$$

where  $n \geq 2$ ,  $t \in [2, \infty)$ . Here we have:

- (i)  $p = \frac{4}{3}$ ,  $a(t) = \frac{1}{t}$ ,  $r(t) = t^3$ ,  $q(t) = t^{-\frac{1}{3}}$ ,  $g(t) = t - 1$ ;
- (ii)  $a'(t) + r(t) = -\frac{1}{t^2} + t^3 > 0$ ,

$$\begin{aligned} \int_2^\infty \left[ \frac{\exp\left(\int_2^s \frac{r(\tau)}{a(\tau)} d\tau\right)}{a(s)} \right]^{\frac{1}{p-1}} ds &= \int_2^\infty \left( se^{\left(\frac{s^5}{5} - \frac{2^5}{5}\right)} \right)^3 ds = e^{-\frac{3 \cdot 2^5}{5}} \int_2^\infty \left( se^{\frac{s^5}{5}} \right)^3 ds \\ &\geq e^{-\frac{3 \cdot 2^5}{5}} \int_2^\infty s^3 ds = \infty; \end{aligned}$$

$$(iii) \int_{t_0}^\infty \exp\left(\int_2^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds = \int_2^\infty e^{\left(\frac{s^5}{5} - \frac{2^5}{5}\right)} s^{-\frac{1}{3}} ds \geq e^{-\frac{2^5}{5}} \int_2^\infty s^{-\frac{1}{3}} ds = \infty.$$

Hence (H1)-(H2), (2.1), and (2.2) hold. For sufficiently large  $T \in [2, \infty)$ , we set  $z(t) = 1$  and have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_T^t \left[ z(s) \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) Q(s) - \frac{z'_+(s)}{\beta_{n-1}^{p-1}(s, T)} \right] ds \\ = \limsup_{t \rightarrow \infty} \int_T^t e^{\left(\frac{s^5}{5} - \frac{2^5}{5}\right)} \left( s^{-\frac{1}{3}} \left( \frac{(s-1)^{n-1}}{s^{n-1}} \right)^{\frac{1}{3}} \right) ds \\ \geq \limsup_{t \rightarrow \infty} e^{-\frac{2^5}{5}} \int_T^t \left( s^{-\frac{1}{3}} \left( \frac{\left(\frac{s}{2}\right)^{n-1}}{s^{n-1}} \right)^{\frac{1}{3}} \right) ds \\ = \limsup_{t \rightarrow \infty} e^{-\frac{2^5}{5}} \int_T^t \frac{1}{2} \frac{1}{s^{\frac{n-1}{3}}} ds = \infty. \end{aligned}$$

Hence Condition (a) of Theorem 2.1 is satisfied.

By Theorem 2.1, every solution  $x(t)$  of (3.1) is oscillatory or tends to 0 as  $t \rightarrow \infty$  when  $n$  is odd, and (3.1) is oscillatory when  $n$  is even.

**Remark 3.1** In Example 1 above, we see that  $a(t) = \frac{1}{t}$  is strictly decreasing on  $[2, \infty)$ ; but in [9], we see that its condition (H1) requires  $a'(t) > 0$ , so the results in [9] are not applicable to this example.

**Example 2** Consider the equation

$$(t|x^{(n-1)}(t)|^2 x^{(n-1)}(t))' - |x^{(n-1)}(t)|^2 x^{(n-1)}(t) + t^2 |x(t+1)|^2 x(t+1) = 0, \quad (3.2)$$

where  $n \geq 2$ ,  $t \in [1, \infty)$ . Here we have:

- (i)  $p = 4$ ,  $a(t) = t$ ,  $r(t) = -1$ ,  $q(t) = t^2$ , and  $g(t) = t + 1$ ;
- (ii)  $a'(t) + r(t) = 1 > 0$ ,

$$\int_1^\infty \left[ \frac{\exp(\int_1^s \frac{r(\tau)}{a(\tau)} d\tau)}{a(s)} \right]^{\frac{1}{p-1}} ds = \int_1^\infty \left( \frac{1}{s} e^{(-\ln s + \ln 1)} \right)^{\frac{1}{3}} ds = \int_1^\infty \left( \frac{1}{s^2} \right)^{\frac{1}{3}} ds = \infty;$$

- (iii)  $\int_{t_0}^\infty \exp(\int_2^s \frac{r(\tau)}{a(\tau)} d\tau) q(s) ds = \int_1^\infty \exp(\int_1^s \frac{-1}{\tau} d\tau) s^2 ds = \int_1^\infty s ds = \infty$ .

Hence (H1)-(H2), (2.1), and (2.2) hold. With  $z(t) = \frac{1}{t}$  we see that for sufficiently large  $T \in [1, \infty)$ ,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_T^t \left[ z(s)Q(s) - \frac{[\frac{z'(s)}{z(s)} - \frac{r(s)}{a(s)}]^2 z(s)}{4(p-1) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) \beta^*(s, T)} \right] ds \\ = \limsup_{t \rightarrow \infty} \int_T^t z(s)Q(s) ds = \limsup_{t \rightarrow \infty} \int_T^t s^2 \cdot \frac{1}{s} ds = \limsup_{t \rightarrow \infty} \int_T^t s ds = \infty. \end{aligned}$$

Hence Condition (a) of Theorem 2.3 is satisfied.

By Theorem 2.3, every solution  $x(t)$  of (3.2) is oscillatory or tends to zero as  $t \rightarrow \infty$  when  $n$  is odd, and (3.2) is oscillatory when  $n$  is even.

**Remark 3.2** In Example 2 above, we see that  $r(t) = -1 < 0$  and  $g(t) = t + 1 \geq t$  on  $[1, \infty)$ . But in [9], we see that its condition (H2) requires  $r(t) > 0$ , and in [14], its condition (H1) requires  $g(t) \leq t$ , so the results in [9, 14] are not applicable to this example.

#### 4 Preliminary lemmas

In this section, we present several technical lemmas which will be used in the proofs of the main results. The first one is on the signs of derivatives of certain classes of functions.

**Lemma 4.1** ([10]) *Let  $f(t) \in C^m([t_0, \infty), (0, \infty))$ . Assume  $f^{(m)}(t)$  is of one sign and not identically zero on  $[t_1, \infty)$  for any  $t_1 \geq t_0$ . Then there exist a  $t_x \geq t_1$  and an integer  $0 \leq l \leq m$ , with  $m + l$  even for  $f^{(m)}(t) \geq 0$  or  $m + l$  odd for  $f^{(m)}(t) \leq 0$  such that*

$$l > 0 \quad \text{implies} \quad f^{(i)}(t) > 0 \quad \text{for } t \geq t_x, i = 0, 1, \dots, l-1 \quad \text{and} \quad (4.1)$$

$$l \leq m-1 \quad \text{implies} \quad (-1)^{l+i} f^{(i)}(t) > 0 \quad \text{for } t \geq t_x, i = l, l+1, \dots, m-1. \quad (4.2)$$

The next lemma concerns the signs of derivatives of eventually positive solutions of (1.1). In particular, we derive conditions for the following inequalities to hold eventually:

$$x^{(j)}(t) > 0, \quad j = 1, 2, \dots, n-1. \quad (4.3)$$

**Lemma 4.2** *Assume that conditions (H1)-(H2), (2.1) and either (2.2) or (2.3) hold. Let  $x(t)$  be an eventually positive solution of (1.1). Then there exists a  $T \in \mathbb{R}$  sufficiently large such that*

$$\left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) |x^{(n-1)}(t)|^{p-2} x^{(n-1)}(t) \right)' \leq 0$$

and  $x^{(n)}(t) < 0$  for  $t \geq T$ . Moreover:

- (i) (4.3) holds when  $n$  is even, and
- (ii) either (4.3) holds or  $\lim_{t \rightarrow \infty} x(t) = 0$  when  $n$  is odd.

*Proof* If  $x(t)$  is an eventually positive solution of (1.1), then by (H2), there exists a  $t_1 \in [t_0, \infty)$  such that

$$x(t) > 0 \quad \text{and} \quad x(g(t)) > 0, \quad t \geq t_1.$$

From (1.1) and (H1), we have

$$\begin{aligned} & \left( a(t) |x^{(n-1)}(t)|^{p-2} x^{(n-1)}(t) \right)' + r(t) |x^{(n-1)}(t)|^{p-2} x^{(n-1)}(t) \\ & = -q(t) |x(g(t))|^{p-2} x(g(t)) \leq 0, \quad t \geq t_1, \end{aligned} \quad (4.4)$$

which implies that

$$\left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) |x^{(n-1)}(t)|^{p-2} x^{(n-1)}(t) \right)' \leq 0, \quad t \geq t_1. \quad (4.5)$$

Then  $a(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) |x^{(n-1)}(t)|^{p-2} x^{(n-1)}(t)$  is strictly decreasing on  $[t_1, \infty)$  and eventually of one sign. So,  $x^{(n-1)}(t)$  is either eventually positive or eventually negative.

By (2.1), from the proof of Lemma 4 in [9], we have  $x^{(n-1)}(t) > 0$ . Then we can write (4.5) in the form

$$\left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) (x^{(n-1)}(t))^{p-1} \right)' \leq 0, \quad t \geq t_1,$$

which implies that for  $t \geq t_1$ ,

$$\begin{aligned} & \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) (a'(t) + r(t)) (x^{(n-1)}(t))^{p-1} \\ & + \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) (p-1) a(t) (x^{(n-1)}(t))^{p-2} x^{(n)}(t) \leq 0. \end{aligned}$$

Thus,  $x^{(n)}(t) < 0$  eventually.

From Lemma 4.1 with  $m = n - 1$ , there exist a  $t_2 \in [t_1, \infty)_{\mathbb{T}}$  and an integer  $l$ ,  $0 \leq l \leq m$  such that (4.1) and (4.2) are satisfied. That is,

$$x^{(j)}(t) > 0 \quad \text{for } t \geq t_2, j = 1, \dots, l-1, \quad (4.6)$$

$$(-1)^{l+j} x^{(j)}(t) > 0 \quad \text{for } t \geq t_2, j = l, l+1, \dots, m-1, \quad (4.7)$$

and  $x(t)$  is eventually monotone.

When  $n$  is even, by Lemma 4.1,  $l$  must be an odd number. By (4.6) and (4.7), we can get  $x'(t) > 0$ . Hence

$$\lim_{t \rightarrow \infty} x(t) \text{ exists and is positive} \quad \text{or} \quad \lim_{t \rightarrow \infty} x(t) = \infty. \quad (4.8)$$

In this case, we claim that  $l = m = n - 1$ . Otherwise, we obtain the odd integer  $l \leq m - 2 = n - 3$ . By (4.7), we get

$$(-1)^{l+m-1} x^{(m-1)}(t) > 0 \quad \text{and} \quad (-1)^{l+m-2} x^{(m-2)}(t) > 0. \quad (4.9)$$

This means

$$x^{(m-1)}(t) = x^{(n-2)}(t) < 0, \quad x^{(m-2)}(t) = x^{(n-3)}(t) > 0, \quad t \in [t_2, \infty). \quad (4.10)$$

It follows from (4.8) that there exist a  $T \geq t_2$  and  $b > 0$  such that  $x(g(t)) \geq b$  for  $t \geq T$ . From (4.4) and (4.5), and  $x^{(n-1)}(t) > 0$  we have

$$\left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) (x^{(n-1)}(t))^{p-1} \right)' \leq -b^{p-1} \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) q(t). \quad (4.11)$$

If (2.2) holds, by integrating (4.11) from  $T$  to  $t$  with  $t \geq T$  we obtain

$$\begin{aligned} & a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) (x^{(n-1)}(t))^{p-1} \\ & \leq a(T) \exp \left( \int_{t_0}^T \frac{r(\tau)}{a(\tau)} d\tau \right) (x^{(n-1)}(T))^{p-1} - b^{p-1} \int_T^t \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) q(s) ds \\ & \rightarrow -\infty \quad \text{as } t \rightarrow \infty, \end{aligned}$$

which contradicts the fact that  $x^{(n-1)}(t) > 0$  for  $t \in [t_1, \infty)$ . Hence,  $l = m = n - 1$  and (4.3) holds. If (2.3) holds, by integrating (4.11) from  $t$  to  $u$  with  $T \leq t \leq u$  we obtain

$$\begin{aligned} & a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) (x^{(n-1)}(t))^{p-1} \\ & \geq a(u) \exp \left( \int_{t_0}^u \frac{r(\tau)}{a(\tau)} d\tau \right) (x^{(n-1)}(u))^{p-1} + b^{p-1} \int_t^u \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) q(s) ds \\ & \geq b^{p-1} \int_t^u \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) q(s) ds. \end{aligned}$$

Taking the limit as  $u \rightarrow \infty$ , we have

$$x^{(n-1)}(t) \geq b \left( a^{-1}(t) \exp \left( - \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) \int_t^\infty \exp \left( \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) q(s) ds \right)^{\frac{1}{p-1}}.$$

Since  $x^{(n-2)}(t) < 0$ , integrating the above inequality from  $t$  to  $v$  with  $T \leq t \leq v$  we get

$$\begin{aligned} -x^{(n-2)}(t) &\geq x^{(n-2)}(v) - x^{(n-2)}(t) \\ &\geq b \int_t^v \left( a^{-1}(s) \exp \left( - \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) \int_s^\infty \exp \left( \int_{t_0}^u \frac{r(\tau)}{a(\tau)} d\tau \right) q(u) du \right)^{\frac{1}{p-1}} ds. \end{aligned}$$

Taking the limit as  $v \rightarrow \infty$ , we obtain

$$-x^{(n-2)}(t) \geq b \int_t^\infty \left( a^{-1}(s) \exp \left( - \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) \int_s^\infty \exp \left( \int_{t_0}^u \frac{r(\tau)}{a(\tau)} d\tau \right) q(u) du \right)^{\frac{1}{p-1}} ds.$$

Since  $x^{(n-3)}(t) > 0$ , integrating the above inequality from  $T$  to  $t$  with  $t \geq T$ , we get

$$\begin{aligned} x^{(n-3)}(T) &\geq -x^{(n-3)}(t) + x^{(n-3)}(T) \\ &\geq b \int_T^t \left[ \int_v^\infty \left( a^{-1}(s) \exp \left( - \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) \int_s^\infty \exp \left( \int_{t_0}^u \frac{r(\tau)}{a(\tau)} d\tau \right) q(u) du \right)^{\frac{1}{p-1}} ds \right] dv. \end{aligned}$$

Taking  $t \rightarrow \infty$ , we obtain

$$\begin{aligned} &\int_T^\infty \left[ \int_v^\infty \left( a^{-1}(s) \exp \left( - \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau \right) \int_s^\infty \exp \left( \int_{t_0}^u \frac{r(\tau)}{a(\tau)} d\tau \right) q(u) du \right)^{\frac{1}{p-1}} ds \right] dv \\ &\leq b^{-1} x^{(n-3)}(T) < \infty, \end{aligned}$$

which contradicts (2.3). Hence,  $l = m = n - 1$  and (4.3) holds.

When  $n$  is odd, by Lemma 4.1,  $l$  must be an even integer. By (4.6) and (4.7), we have either  $x'(t) > 0$  or  $x'(t) < 0$ . That means  $\lim_{t \rightarrow \infty} x(t) = c \geq 0$ . We claim that if  $\lim_{t \rightarrow \infty} x(t) \neq 0$ , then  $l = m = n - 1$ . Otherwise, there is the even number  $l \leq m - 2 = n - 3$  such that (4.9) and (4.10) hold. By a similar argument to above, we can reach a contradiction to (2.2) or (2.3). This completes the proof.  $\square$

**Lemma 4.3** *Let conditions (H1)-(H2), (2.1), and either (2.2) or (2.3) hold. Assume  $x(t)$  is an eventually positive solution of (1.1) satisfying (4.3) eventually. Then there exists a  $T \in [t_0, \infty)$  such that for  $t \in [T, \infty)$ ,*

$$x'(t) \geq \left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \beta_{n-2}(t, T)$$

and

$$x(t) \geq \left( a(t) \exp \left( \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau \right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \beta_{n-1}(t, T).$$



*Proof* Since  $x(t)$  is an eventually positive solution of (1.1), by (H2), there exists a  $T \in [t_0, \infty)$  such that  $x(t) > 0$ ,  $x(g(t)) > 0$ , and (4.3) holds for  $t \geq T$ .

Noting that  $(a(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) (x^{(n-1)}(t))^{p-1})' < 0$  for  $t \geq T$ , it follows from (4.3) that

$$\begin{aligned} x^{(n-2)}(t) &= x^{(n-2)}(T) + \int_T^t \frac{(a(s) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) (x^{(n-1)}(s))^{p-1})^{\frac{1}{p-1}}}{(a(s) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau))^{\frac{1}{p-1}}} ds \\ &\geq \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \int_T^t \frac{1}{(a(s) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau))^{\frac{1}{p-1}}} ds \\ &:= \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \beta_1(t, T). \end{aligned}$$

Integrating the inequality above from  $T$  to  $t$  for  $t \geq T$ , we get

$$\begin{aligned} x^{(n-3)}(t) &= x^{(n-3)}(T) + \int_T^t x^{(n-2)}(s) ds \\ &\geq x^{(n-3)}(T) + \int_T^t \left( a(s) \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(s) \beta_1(s, T) ds \\ &\geq \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \int_T^t \beta_1(s, T) ds \\ &:= \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \beta_2(t, T). \end{aligned}$$

It is easy to see by induction that

$$\begin{aligned} x'(t) &\geq \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \beta_{n-2}(t, T), \\ x(t) &\geq \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) \right)^{\frac{1}{p-1}} x^{(n-1)}(t) \beta_{n-1}(t, T). \end{aligned}$$

This completes the proof.  $\square$

Similar to [16], Lemma 2.3, we have the following lemma.

**Lemma 4.4** Let  $g(u) = Bu - Au^{\frac{\gamma+1}{\gamma}}$ , where  $A$  and  $\gamma$  are positive numbers, and  $B \geq 0$ . Then  $g$  attains its maximum value on  $[0, \infty)$  at  $u^* = (\frac{B\gamma}{A(\gamma+1)})^\gamma$ , and

$$\max_{u \in [0, \infty)} g = g(u^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}.$$

## 5 Proofs of main results

In this section, we give proofs for our main results by employing generalized Riccati techniques and integral averaging techniques.

*Proof of Theorem 2.1* Suppose to the contrary that (1.1) has a nonoscillatory solution  $x(t)$ . Without loss of generality, we may assume that  $x(t)$  is eventually positive. Then, by (H1)-(H2), there exists a  $T \in [t_0, \infty)$  such that for  $t \geq T$ ,  $x(t) > 0$ , and Lemmas 4.2 and 4.3 hold.

When  $n$  is odd, from Lemma 4.2 we see that (4.3) holds or  $\lim_{t \rightarrow \infty} x(t) = 0$ . If (4.3) holds, (1.1) reduces to

$$(a(t)(x^{(n-1)}(t))^{p-1})' + r(t)(x^{(n-1)}(t))^{p-1} + q(t)(x(g(t)))^{p-1} = 0. \quad (5.1)$$

Multiplying by  $\exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)$  on (5.1), we have

$$\left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) (x^{(n-1)}(t))^{p-1} \right)' + \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) q(t)(x(g(t)))^{p-1} = 0. \quad (5.2)$$

The rest of the proof is separated into three parts corresponding to conditions (a)-(c), respectively.

Part I: Assume condition (a) holds. Define

$$v(t) := \frac{z(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) a(t) (x^{(n-1)}(t))^{p-1}}{x^{p-1}(t)} \quad \text{for } t \geq T.$$

Then  $v(t) > 0$ . From  $z(t) > 0$ ,  $a(t) > 0$ ,  $x(t) > 0$ ,  $x^{(n-1)}(t) > 0$ , and (5.2), we have

$$\begin{aligned} v'(t) &= \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) (x^{(n-1)}(t))^{p-1} \right)' \left( \frac{z(t)}{x^{p-1}(t)} \right) \\ &\quad + \left( \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) a(t) (x^{(n-1)}(t))^{p-1} \right) \left( \frac{z(t)}{x^{p-1}(t)} \right)' \\ &= z(t) \frac{-\exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) q(t) (x(g(t)))^{p-1}}{x^{p-1}(t)} \\ &\quad + \left( a(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) (x^{(n-1)}(t))^{p-1} \right) \left[ \frac{z'(t) x^{p-1}(t) - z(t) (n-1) x^{p-2}(t) x'(t)}{(x^{p-1}(t))^2} \right] \\ &\leq \frac{-z(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) q(t) (x(g(t)))^{p-1}}{x^{p-1}(t)} \\ &\quad + \frac{z'_+(t) (a(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) (x^{(n-1)}(t))^{p-1})}{x^{n1}(t)} \\ &\quad - \frac{(a(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) (x^{(n-1)}(t))^{p-1}) z(t) (p-1) x^{p-2}(t) x'(t)}{(x^{p-1}(t))^2} \\ &\leq \frac{-z(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) q(t) (x(g(t)))^{n1}}{x^{p-1}(t)} + \frac{z'_+(t) (a(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) (x^{(n-1)}(t))^{p-1})}{x^{p-1}(t)}. \end{aligned}$$

By the Kiguradze lemma [8], which shows that if a function  $y(t)$  satisfies  $y^{(i)}(t) > 0$ ,  $i = 0, 1, 2, \dots, k$ , and  $y^{(k+1)}(t) \leq 0$ , then  $y(t)/y'(t) \geq t/k$ , we have

$$\frac{x(t)}{x'(t)} \geq \frac{t}{n-1}.$$

Thus, we see that  $x(t)/t^{n-1}$  is nonincreasing, and when  $g(t) \leq t$ , we get

$$\frac{x(g(t))}{x(t)} \geq \frac{g^{n-1}(t)}{t^{n-1}}.$$

When  $g(t) > t$ , by  $x'(t) > 0$ , we get

$$\frac{x(g(t))}{x(t)} \geq 1.$$

It follows that

$$\left( \frac{x(g(t))}{x(t)} \right)^{p-1} \geq G(t). \quad (5.3)$$

Then, by Lemma 4.3, we obtain

$$v'(t) \leq -z(t) \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) q(t) G(t) + \frac{z'_+(t)}{\beta_{n-1}^{p-1}(t, T)}.$$

Integrating the above inequality from  $T$  to  $t$  for  $t \geq T$ , we get

$$\int_T^t \left[ z(s) \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) Q(s) - \frac{z'_+(s)}{\beta_{n-1}^{n1}(s, T)} \right] ds \leq v(T) - v(t) < v(T).$$

Taking lim sup on both sides as  $t \rightarrow \infty$ , we obtain a contradiction to condition (b). Therefore, every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$ .

Part II: Assume condition (b) holds. Define

$$w(t) := \frac{z(t)a(t)(x^{(n-1)}(t))^{n1}}{x^{p-1}(t)} \quad \text{for } t \geq T. \quad (5.4)$$

Then  $w(t) > 0$ . From  $z(t) > 0$ ,  $x(t) > 0$ ,  $x^{(n-1)}(t) > 0$ , we have

$$\begin{aligned} w'(t) &= (a(t)(x^{(n-1)}(t))^{p-1})' \left( \frac{z(t)}{x^{p-1}(t)} \right) + (a(t)(x^{(n-1)}(t))^{n1}) \left( \frac{z(t)}{x^{p-1}(t)} \right)' \\ &= z(t) \frac{-r(t)(x^{(n-1)}(t))^{p-1} - q(t)(x(g(t)))^{p-1}}{x^{p-1}(t)} \\ &\quad + (a(t)(x^{(n-1)}(t))^{p-1}) \left[ \frac{z'(t)x^{p-1}(t) - z(t)(n1)x^{p-2}(t)x'(t)}{(x^{p-1}(t))^2} \right] \\ &\leq \frac{-z(t)r(t)(x^{(n-1)}(t))^{p-1} - z(t)q(t)(x(g(t)))^{p-1}}{x^{n1}(t)} + \frac{z'_+(t)(a(t)(x^{(n-1)}(t))^{p-1})}{x^{p-1}(t)} \\ &\quad - \frac{(a(t)(x^{(n-1)}(t))^{p-1})z(t)(p-1)x^{p-2}(t)x'(t)}{(x^{n1}(t))^2}. \end{aligned} \quad (5.5)$$

From (5.3), we have

$$\begin{aligned} w'(t) &\leq -z(t)Q(t) + \left[ \frac{z'_+(t)}{z(t)} - \frac{r(t)}{a(t)} \right] w(t) \\ &\quad - \frac{(a(t)(x^{(n-1)}(t))^{n1})z(t)(p-1)x^{p-2}(t)x'(t)}{(x^{p-1}(t))^2}. \end{aligned} \quad (5.6)$$

From Lemma 4.3, we get

$$\begin{aligned} & -\frac{(a(t)(x^{(n-1)}(t))^{n1})z(t)(p-1)x^{p-2}(t)x'(t)}{(x^{p-1}(t))^2} \\ & = -\frac{a(t)z(t)(p-1)}{a^{\frac{p}{p-1}}(t)z^{\frac{p}{p-1}}(t)} \frac{a^{\frac{p}{n1}}(t)z^{\frac{p}{p-1}}(t)(x^{(n-1)}(t))^p}{x^p(t)} \frac{x'(t)}{x^{(n-1)}(t)} \\ & \leq -\frac{(p-1)\beta_{n-2}(t, T) \exp(\frac{1}{p-1} \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(t)} \frac{a^{\frac{p}{p-1}}(t)z^{\frac{p}{n1}}(t)(x^{(n-1)}(t))^p}{x^p(t)}. \end{aligned}$$

Setting  $\gamma = p - 1$ , we have

$$\begin{aligned} w'(t) & \leq -z(t)Q(t) + \left[ \frac{z'_+(t)}{z(t)} - \frac{r(t)}{a(t)} \right] w(t) \\ & \quad - \frac{(p-1)\beta_{n-2}(t, T) \exp(\frac{1}{p-1} \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(t)} w^{\frac{\gamma+1}{\gamma}}(t) \\ & \leq -z(t)Q(t) + \left[ \frac{z'_+(t)}{z(t)} - \frac{r(t)}{a(t)} \right]_+ w(t) \\ & \quad - \frac{(p-1)\beta_{n-2}(t, T) \exp(\frac{1}{n1} \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(t)} w^{\frac{\gamma+1}{\gamma}}(t). \end{aligned} \quad (5.7)$$

Let

$$B = \left[ \frac{z'_+(t)}{z(t)} - \frac{r(t)}{a(t)} \right]_+, \quad A = \frac{(p-1)\beta_{n-2}(t, T) \exp(\frac{1}{p-1} \int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(t)}, \quad u = w(t).$$

Then by Lemma 4.4, we obtain, for all  $t \geq T$ ,

$$w'(t) \leq -z(t)Q(t) + \frac{1}{p^p} \left[ \frac{z'_+(t)}{z(t)} - \frac{r(t)}{a(t)} \right]_+^p \frac{z(t)}{\beta_{n-2}^{p-1}(t, T) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)}.$$

Integrating the above inequality from  $T$  to  $t$  for  $t \geq T$ , we get

$$\begin{aligned} & \int_T^t \left[ z(s)Q(s) - \frac{1}{p^p} \left[ \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right]_+^p \frac{z(s)}{\beta_{n-2}^{p-1}(s, T) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau)} \right] ds \\ & \leq w(T) - w(t) < w(T). \end{aligned}$$

By taking limsup on both sides as  $t \rightarrow \infty$ , we obtain a contradiction to condition (c).

Therefore, every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$ .

Part III: Assume condition (c) holds. From (5.7) we have, for  $H \in \mathcal{H}_*$  and  $t \geq T$ ,

$$\begin{aligned} \int_T^t H(t, s)z(s)Q(s) ds & \leq - \int_T^t H(t, s)w'(s) ds + \int_T^t H(t, s)w(s) \left[ \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right] ds \\ & \quad - \int_T^t H(t, s) \frac{(p-1)\beta_{n-2}(s, T) \exp(\frac{1}{p-1} \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(s)} w^{\frac{\gamma+1}{\gamma}}(s) ds. \end{aligned}$$

By integration by parts we obtain

$$-\int_T^t H(t,s)w'(s)ds = H(t,T)w(T) + \int_T^t H'_s(t,s)w(s)ds.$$

It follows that

$$\begin{aligned} \int_T^t H(t,s)z(s)Q(s)ds &\leq H(t,T)w(T) + \int_T^t \left[ H'_s(t,s) + H(t,s) \left( \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right) \right]_+ w(s)ds \\ &\quad - \int_T^t H(t,s) \frac{(p-1)\beta_{n-2}(s,T) \exp(\frac{1}{p-1} \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(s)} w^{\frac{\gamma+1}{\gamma}}(s)ds. \end{aligned}$$

Let  $\gamma = p-1$ ,  $u = w(s)$  and

$$\begin{aligned} B &= \left[ H'_s(t,s) + H(t,s) \left( \frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)} \right) \right]_+, \\ A &= H(t,s) \frac{(p-1)\beta_{n-1}(s,T) \exp(\frac{1}{p-1} \int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau)}{z^{\frac{1}{p-1}}(s)}, \end{aligned}$$

by Lemma 4.4 we obtain, for all  $t \geq T$ ,

$$\int_T^t H(t,s)z(s)Q(s)ds \leq H(t,T)w(T) + \int_T^t \frac{[H'_s(t,s) + H(t,s)(\frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)})]_+^p z(s)}{\beta_{n-2}^{p-1}(s,T)H^{p-1}(t,s)p^p \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau)} ds.$$

That is,

$$\frac{1}{H(t,T)} \int_T^t \left[ H(t,s)z(s)Q(s) - \frac{[H'_s(t,s) + H(t,s)(\frac{z'_+(s)}{z(s)} - \frac{r(s)}{a(s)})]_+^p z(s)}{p^p \beta_{n-2}^{p-1}(s,T)H^{p-1}(t,s) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau)} \right] ds \leq w(T).$$

By taking limsup on both sides as  $t \rightarrow \infty$ , we obtain a contradiction to condition (d). Therefore, every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$ .

When  $n$  is even, from Lemma 4.2 we see that only (4.3) holds. Similarly, we can show that (1.1) is oscillatory. We omit the details.

The proof is complete.  $\square$

**Proof of Theorem 2.2** Suppose to the contrary that (1.1) has a nonoscillatory solution  $x(t)$ . Without loss of generality, we may assume that  $x(t)$  is eventually positive. Then, by (H1)-(H2), there exists a  $T \in [t_0, \infty)$  such that for  $t \geq T$ ,  $x(t) > 0$ , and Lemmas 4.2 and 4.3 hold.

When  $n$  is odd, from Lemma 4.2 we see that (4.3) holds or  $\lim_{t \rightarrow \infty} x(t) = 0$ . If (4.3) holds, we set  $\phi(t) := a(t) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) (x^{(n-1)}(t))^{p-1}$ . Then  $\phi(t) > 0$  and  $\phi'(t) < 0$  for  $t \geq T$ , and  $\lim_{t \rightarrow \infty} \phi(t) = \zeta \geq 0$ . By (5.2), it follows that

$$\phi'(t) + \exp\left(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau\right) q(t)(x(g(t)))^{p-1} = 0. \quad (5.8)$$

Integrating both sides of (5.8) from  $t$  to  $\infty$ , we obtain

$$\zeta - \phi(t) + \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s)(x(g(s)))^{p-1} ds = 0. \quad (5.9)$$

When  $g(t) \geq t$ , we have

$$\zeta - \phi(t) + (x(t))^{p-1} \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds < 0,$$

by which we have reached a contradiction if  $\int_t^\infty \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) q(s) ds = \infty$ . If  $\int_t^\infty \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) q(s) ds < \infty$ , we get

$$\phi(t) \geq (x(t))^{p-1} \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds.$$

By Lemma 4.3, we obtain

$$\begin{aligned} \beta_*^{p-1}(t, T) \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds \\ \leq \beta_{n-1}^{p-1}(t, T) \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds \leq 1, \end{aligned}$$

which is a contradiction to (2.4).

When  $g(t) \leq t$  and  $g'(t) > 0$ , by (5.9), we get

$$\zeta - \phi(t) + (x(g(t)))^{p-1} \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds < 0,$$

by which we have reached a contradiction if  $\int_t^\infty \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) q(s) ds = \infty$ . If  $\int_t^\infty \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) q(s) ds < \infty$ , we get

$$\phi(g(t)) \geq \phi(t) \geq (x(g(t)))^{p-1} \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds.$$

By Lemma 4.3, we obtain

$$\begin{aligned} \beta_*^{p-1}(t, T) \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds \\ \leq \beta_{n-1}^{p-1}(g(t), T) \int_t^\infty \exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau\right) q(s) ds \leq 1, \end{aligned}$$

which is a contradiction to (2.4). Therefore, every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$ .

When  $n$  is even, from Lemma 4.2 we see that only (4.3) holds. Similarly, we can show that (1.1) is oscillatory. We omit the details.

The proof is complete.  $\square$

*Proof of Theorem 2.3* Suppose to the contrary that (1.1) has a nonoscillatory solution  $x(t)$ . Without loss of generality, we may assume that  $x(t)$  is eventually positive. Then, by (H1)-(H2), there exists a  $T \in [t_0, \infty)$  such that for  $t \geq T$ ,  $x(t) > 0$ ,  $x(g(t)) > 0$ , and Lemmas 4.2 and 4.3 hold.

When  $n$  is odd, from Lemma 4.2 we see that either (4.3) holds or  $\lim_{t \rightarrow \infty} x(t) = 0$ . If (4.3) holds, the rest of the proof is separated into two parts corresponding to conditions (a) and (b), respectively.

Part I: Assume condition (a) holds.

Define  $w(t)$  as in (5.4). By  $x(t) > 0$ , and (5.6) we obtain

$$\begin{aligned} w'(t) &\leq -z(t)Q(t) + \left[ \frac{z'(t)}{z(t)} - \frac{r(t)}{a(t)} \right] w(t) - \frac{(a(t)z(t)(x^{(n-1)}(t))^{p-1})^2}{(x^{p-1}(t))^2} \frac{(p-1)x^{p-2}(t)x'(t)}{z(t)a(t)(x^{(n-1)}(t))^{p-1}} \\ &\leq -z(t)Q(t) + \left[ \frac{z'(t)}{z(t)} - \frac{r(t)}{a(t)} \right] w(t) - \frac{(p-1)x^{p-2}(t)}{z(t)a(t)(x^{(n-1)}(t))^{p-2}} \frac{x'(t)}{x^{(n-1)}(t)} (w(t))^2. \end{aligned}$$

From this and Lemma 4.3, we get

$$w'(t) \leq -z(t)Q(t) + w(t) \left[ \frac{z'(t)}{z(t)} - \frac{r(t)}{a(t)} \right] - \frac{(p-1) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau)}{z(t)} \beta^*(t, T) (w(t))^2. \quad (5.10)$$

By completing the square for  $w(t)$  on the right-hand side, we have

$$w'(t) \leq -z(t)Q(t) + \frac{[\frac{z'(t)}{z(t)} - \frac{r(t)}{a(t)}]^2 z(t)}{4(p-1) \exp(\int_{t_0}^t \frac{r(\tau)}{a(\tau)} d\tau) \beta^*(t, T)}.$$

Integrating the above inequality from  $T$  to  $t$  for  $t \geq T$ , we get

$$\int_T^t \left[ z(s)Q(s) - \frac{[\frac{z'(s)}{z(s)} - \frac{r(s)}{a(s)}]^2 z(s)}{4(p-1) \exp(\int_{t_0}^s \frac{r(\tau)}{a(\tau)} d\tau) \beta^*(s, T)} \right] ds \leq w(T) - w(t) < w(T).$$

Taking  $\limsup$  on both sides as  $t \rightarrow \infty$ , we obtain a contradiction to condition (a). Therefore, every solution  $x(t)$  of (1.1) is either oscillatory or tends to zero as  $t \rightarrow \infty$ .

Part II: Assume condition (b) holds.

Based on (5.10), the proof is similar to those of Part III of Theorem 2.1 and Part I of Theorem 2.3, and hence it is omitted.

When  $n$  is even, from Lemma 4.2 we see that only (4.3) holds. Similarly, we can show that (1.1) is oscillatory and hence omit its proof.

The proof is complete.  $\square$

## 6 Conclusions

In this paper, we have discussed the asymptotics and oscillation for  $n$ th-order nonlinear differential equation (1.1), where the deviation argument  $g(t)$  may be retarded, advanced, or mixed. Under certain assumptions, we have derived a complete characterization of an eventually positive solution  $x(t)$  of (1.1): there exists  $T \in \mathbb{R}$  such that for all  $t \geq T$ ,  $(a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t))' \leq 0$ ,  $x^{(n)}(t) < 0$ , and

- (i)  $x^{(j)}(t) > 0$  ( $j = 1, 2, \dots, n-1$ ) when  $n$  is even;
- (ii) either  $x^{(j)}(t) > 0$  ( $j = 1, 2, \dots, n-1$ ) holds or  $\lim_{t \rightarrow \infty} x(t) = 0$  when  $n$  is odd.

By using generalized Riccati techniques and integral averaging techniques, we have proved that under a number of conditions:

- (i) every solution  $x(t)$  of (1.1) is oscillatory or tends to zero as  $t \rightarrow \infty$  when  $n$  is odd;
- (ii) equation (1.1) is oscillatory when  $n$  is even.

Also, we have given two examples to illustrate the obtained results.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors completed the paper together. All authors read and approve the final manuscript.

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